LESSON 23

# Integration Formulas

These integration formulas follow as an extension of our rules for differentiation.

Fundamental integration formulas

1. $\int du = u + C$
2. $\int (du + dr + .... + dv) = \int du + \int dr + \int . . . . + \int dv$
3. $\int c\, du = c \int du$ where c is a constant
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int du/u = \ln u + C$
6. $\int e^u du = e^u + C$
7. $\int \cos(u)\, du = \sin(u) + C$
8. $\int \sin(u)\, du = -\cos(u) + C$
9. $\int \sec^2(u)\, du = \tan(u) + C$
10. $\int \csc^2(u)\, du = -\cot(u) + C$
11. $\int \sec(u) \tan(u)\, du = \sec(u) + C$
12. $\int \csc(u) \cot(u)\, du = -\csc(u) + C$

We can now employ the substitution method of integration.

**Example 1**

$\int 2\cos(2X)\, dX$ Let u = 2X $\frac{du}{dX} = 2$ du = 2dX

Substituting we get :

$\int \cos(u) \cdot dX = \int \cos(u)\, du$

$= \sin(u) + C = \sin(u) + C$

Let's check our work: $d/_{dX}(\sin(2X)) = \cos(2X) \cdot 2$

**Example 2**

$\int 3\sec(3X)\tan(3X)\, dX$ Let u = 3X $\frac{du}{dX} = 3$ du = 3dX

Substituting $\int \sec(3X)\tan(3X) \cdot u$

$\int \sec(u) \cdot \tan(u)\, du = \sec(u) + C =$ $\sec(u(3X)) + C$

Check: $d/_{dX}(\sec(3X)) = \sec(3X)\tan(3X) \cdot 3$

What if the constant I need is not present?

**Example 3**

$\int \sec^2(2X)\,dX$ Let $u = 2X$ $\frac{du}{dX} = 2$ $du = 2dX$

This problem is missing the 2 needed for a complete du value. It can be inserted in the following way. The needed value is placed in the integral and the reciprocal is placed outside. This is the same as multiplication by 1 and does not change the value of the integral.

$$\frac{1}{2}\int \sec^2(2X)\cdot dX\cdot 2$$

$$\frac{1}{2}\int \sec^2(u)du = \frac{1}{2}\tan(u) + C$$

$$= \frac{1}{2}\tan(2X) + C$$

Check: $d/dX\ (\frac{1}{2}\tan(2X) = \frac{1}{2}\sec^2(2X)\cdot 2 = \sec^2(2X)$

**Example 4**

$\int 2X(1 + X^2)^8 dX$

You can use this same idea with polynomials. You could use Pascal's triangle to write out all the terms, but let's try substitution.

Let $u = 1 + X^2$ $du = 2XdX$ $\frac{du}{dX} = 2X$

Rearranging we get

$$\int (1 + X^2)^8 2XdX$$

$$\int u^8 du = \frac{u^9}{9} + C = \frac{1}{9}(1 + X^2)^9 + C$$

Check: $d/dX\ (\frac{1}{9}(1+X^2)^9) = \frac{1}{9}\cdot 9(1 + X^2)^8\cdot 2X = 2X(1 + X^2)^8$

**Example 5**

$\int X(1 - X^2)^7 dX$

$u = 1 - X^2 \quad \frac{du}{dX} = -2X \qquad du = -2XdX$

This time we do not have the needed constant. Multiplying by one, we get

$$-\frac{1}{2}\int -2X\,(1-X^2)^7 dX = -\frac{1}{2}\int (1-X^2)^7(-2XdX)$$

$$= -\frac{1}{2}\int u^7 du = -\frac{1}{2}\;\frac{u^8}{8} + C$$

$$= -\frac{1}{16}\;(1 - X^2)^8 + C$$

Check: $d/_{du}\,(-\frac{1}{16}(1-X^2)^8) = -\frac{1}{16}\cdot 8(1 - X^2)^7(-2X)$

$$= X(1-X^2)^7$$

The tricky part will be determining what u will be. If you pick something that is incorrect for u, then try again. These problems will become simpler with practice.

**Example 6**

$\int e^{5X} dX \qquad u = 5X \qquad \frac{du}{dX} = 5 \qquad du = 5dX$

Multiply by 1

$$\frac{1}{5}\int 5e^u du = \frac{1}{5}\;e^u + C = \frac{1}{5}\;e^{5X} + C$$

**Example 7**

$$\int \frac{\cos(2\theta)}{\sin^2(2\theta)} \cdot 5d\theta$$

$$u = (\sin(2\theta))$$

$$\frac{du}{d\theta} = \cos(2\theta) \cdot 2$$

$$du = \cos(2\theta) \cdot 2d\theta$$

Here we have an example where we have the wrong constant. Simply move the wrong constant out in front of the integral and supply the correct constant by multiplying by 1.

$$5\int \frac{\cos(2\theta)}{\sin(2\theta)^2} \cdot d\theta = \frac{1}{2} \cdot 5\int \frac{\cos(2\theta)}{\sin(2\theta)^2} \cdot 2d\theta$$

$$= \frac{5}{2} \int \frac{du}{u^2} = \frac{5}{2} \int u^{-2} du = \frac{5}{2} \frac{u^{-1}}{-1} + C$$

$$= -\frac{5}{2} u^{-1} + C = \frac{-5}{2(\sin(2\theta))} + C$$

**Example 8**

$$\int \frac{(X^2-1)^2}{X^2} dX$$

None of the formulas we have learned cover this form of the integral. Thankfully the exponent is small. We can square the numerator and divide by the denominator.

$$\int \frac{(X^2-1)^2}{X^2} dX = \int \left(\frac{X^4-2X^2+1}{X^2}\right) dX$$

$$\int X^2 dX - \int 2dX + \int X^{-2} dX$$

$$= \frac{X^3}{3} - 2X + \frac{X^{-1}}{-1} + C$$

$$= \frac{1}{3}X^3 - 2X - \frac{1}{X} + C$$

Check : $\frac{d}{dX}\left(\frac{1}{3}X^3 - 2X - \frac{1}{X}\right) =$

$$\frac{1}{3}(3X^2) - 2 - (-X^{-2})$$

$$= X^2 - 2 + \frac{1}{X^2}$$

$$= \frac{X^4-2X^2+1}{X^2} = \frac{(X^2-1)^2}{X^2}$$