

LESSON 3

Hyperbola and Systems of Equations

HYPERBOLAS

We will begin our discussion of hyperbolas with left-right hyperbolas. The X^2 and Y^2 terms of circles and ellipses are both positive. What distinguishes a hyperbola is that one component is negative. The equation $x^2 + y^2 = 5^2$ is a circle and $x^2 + 2y^2 = 5^2$ is an ellipse, but $x^2 - y^2 = 5$ or $y^2 - x^2 = 2$ is a hyperbola. The standard form of a hyperbola centered at the origin is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertices are $(\pm a, 0)$. Asymptotes are $y = \pm \frac{b}{a}x$.

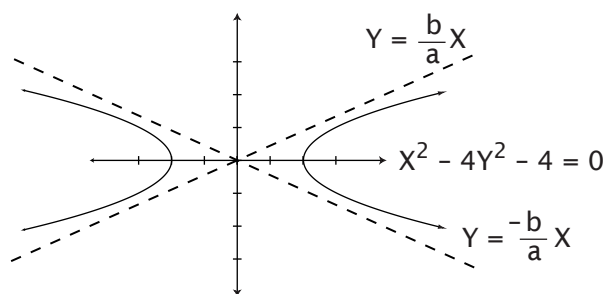
Example 1

$$x^2 - 4y^2 - 4 = 0$$

$$x^2 - 4y^2 = 4$$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

$$\frac{x^2}{2^2} - \frac{y^2}{1^2} = 1$$



The vertices are $(2, 0)$ and $(-2, 0)$.

The asymptotes are $y = \pm \frac{1}{2}x$.

The up-down or north-south hyperbolas are quite similar.

The standard form centered at the origin is:

$$\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$$

Vertices are $(0, \pm a)$. The asymptotes are $y = \pm \frac{a}{b}x$.

The "a" in the denominator, is always paired with the positive component. In the standard form of a hyperbola on page 35 a^2 is under x^2 . In this example, a^2 is under y^2 . When finding the asymptote using a and b, the letter that accompanies y is always the numerator since slope is computed by the rise (y-component) over the run (x-component)!

Example 2

$$-4X^2 + Y^2 - 4 = 0$$

$$Y^2 - 4X^2 = 4$$

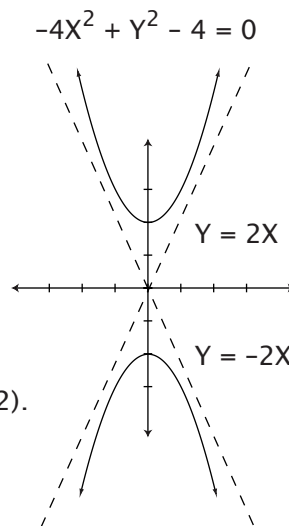
$$\frac{Y^2}{4} - \frac{X^2}{1} = 1$$

$$a = 2 \quad b = 1$$

$$h = 0 \quad k = 0$$

The vertices are $(0, 2)$ and $(0, -2)$.

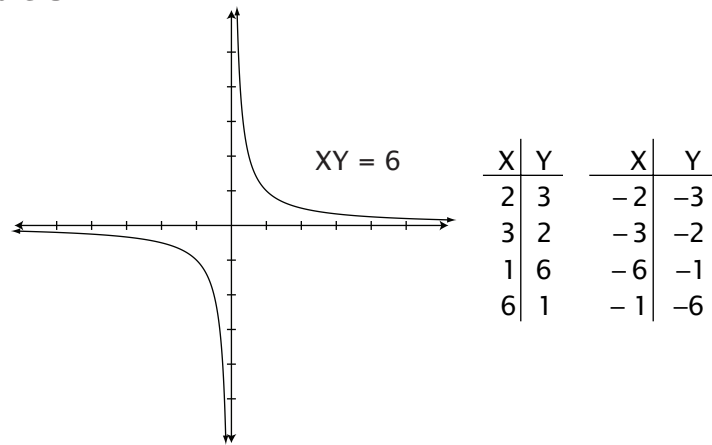
The asymptotes are $y = \pm 2x$.



You may have noticed that rearranging the equation will give you more information which helps to draw a graph.

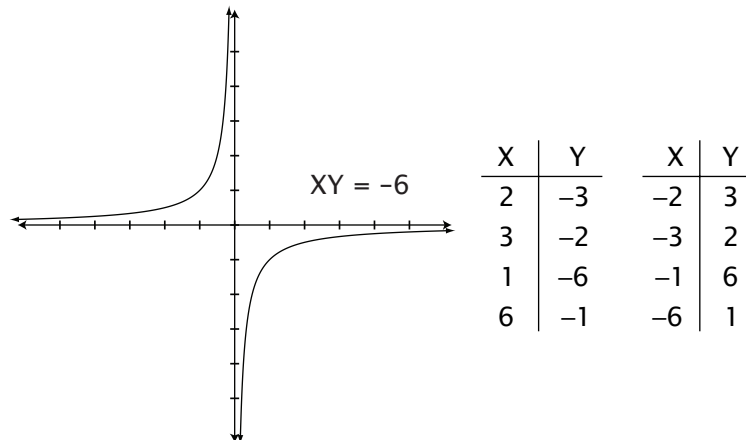
Lastly, there are hyperbolas which have the equation $xy = +c$ or $xy = -c$.

Example 3



Plotting points will quickly yield a graph in quadrants I and III.

Example 4



The graph of $xy = -6$ yields a graph in quadrants II and IV.

SYSTEMS OF EQUATIONS

Especially in the second semester of Calculus, we will be learning about areas between curves and we will need to solve systems of equations and graph them.

Example 5

Find the intersection of $Y = 1 - X^2$ and $Y = X^2 - 1$.

These are parabolas (one squared term). They both have a vertical axis of symmetry ($x = 0$). The first is a “frown” and the second is a “smile.”

$$1 - X^2 = X^2 - 1$$

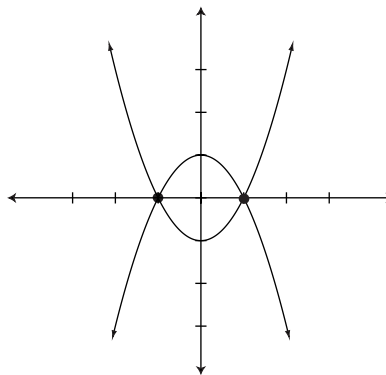
$$2 = 2X^2$$

$$1 = X^2$$

$$X = \pm 1$$

Plugging into either of the original equations, we get $y = 1 - 1 = 0$.

Intersection is $(\pm 1, 0)$



Example 6

Find the intersection of $Y = X^2$ and $X - Y + 2 = 0$.

$$X^2 = X + 2$$

$$X^2 - X - 2 = 0$$

$$(X + 1)(X - 2) = 0$$

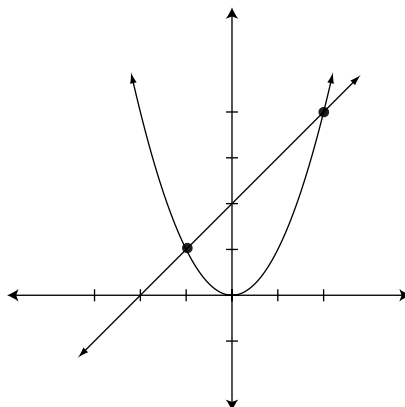
$$X = -1, 2$$

Substituting into $y = x^2$

$$Y = (-1)^2 = 1$$

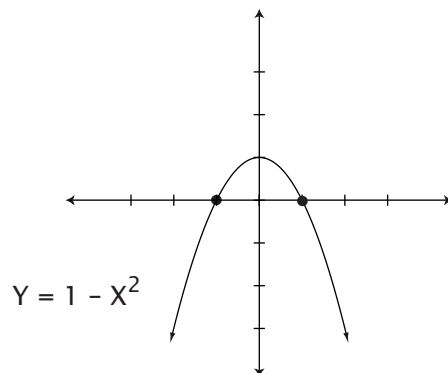
$$Y = 2^2 = 4$$

Intersection $(-1, 1) (2, 4)$



INEQUALITIES

Let's revisit example 5. Where is $Y \geq 1 - X^2$? First, we graph $Y = 1 - X^2$. Then we plug in arbitrary values for X and Y and determine whether or not those values are above or below the curve.



Let's try (0,0). Plugging into $Y \geq 1 - X^2$ we get

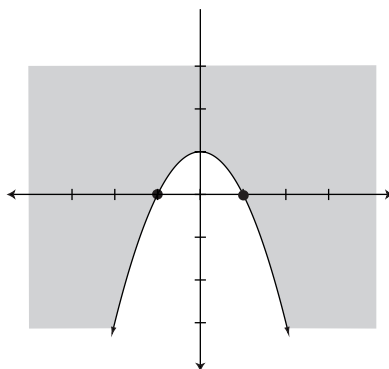
$$0 > 1 - 0^2 \text{ or}$$

$0 > 1$ which is false, so (0, 0) is not a solution.

Now try (1, 4). Plugging into $Y \geq 1 - X^2$ we get

$$4 \geq 1 - 1^2 \text{ or}$$

$4 \geq 0$ which is true. We can now, with confidence shade our solutions which include (1, 4) and exclude (0, 0)



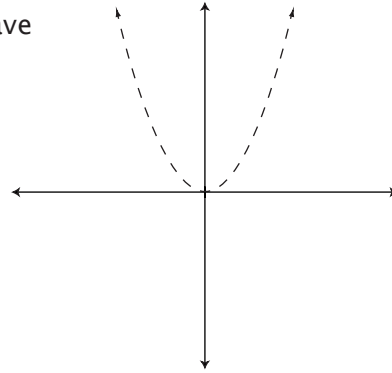
Example 7

Shade the region which represents all solutions to $Y < X^2$.

Drawing $Y = X^2$ with dotted lines we have

Picking $(0, 2)$ we get

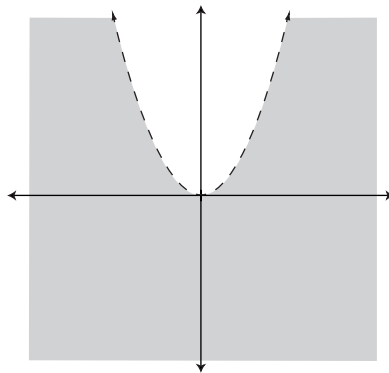
$2 < 0^2$ which is false so $(0, 2)$
is not a solution



Picking $(0, -2)$ we get

$-2 < 0^2$
 $-2 < 0$ which is true

Our answer is



All quadratic equations follow the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Curve	A	B	C	D	E	F	Comments
Line	0	0	0	RN	RN	RN	D = 0 yields a horizontal line
							E = 0 yields a vertical line
Parabola (up/down)	RN	0	0	R	RN	R	A, E same signs = "frown"
							A, E different signs = "smile"
Parabola (left/right)	0	0	RN	RN	R	R	C, D same sign = "c" on left
							C, D different sign = "c" on right
Circle	RN	0	RN	R	R	RN	A must equal C
Ellipse	RN	0	RN	R	R	RN	A, C must be the same sign
							x > y = football, y > x = egg
Hyperbola (left/right)	RN	0	RN	R	R	RN	+A = symmetry about y axis
							A & C must be opposite signs
Hyperbola (up/down)	RN	0	RN	R	R	RN	-A = symmetry about y axis
							A & C must be opposite signs
Hyperbola (quadrants)	0	RN	0	0	0	RN	+ F = graph in quadrants II, IV
							- F = graph in quadrants I, III

0 - means that the value of the coefficient must be zero.

R - the value of the coefficient may be any positive or negative real number, including zero.

RN - the value of the coefficient may be any positive or negative real number, not including zero.